

# New Light on Infrared Problems: Sectors, Statistics, Spectrum and All That\*

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## Abstract

Within the general setting of algebraic quantum field theory, a new approach to the analysis of the physical state space of a theory is presented; it covers theories with long range forces, such as quantum electrodynamics. Making use of the notion of charge class, which generalizes the concept of superselection sector, infrared problems are avoided. In fact, on this basis one can determine and classify in a systematic manner the proper charge content of a theory, the statistics of the corresponding states and their spectral properties. A key ingredient in this approach is the fact that in real experiments the arrow of time gives rise to a Lorentz invariant infrared cutoff of a purely geometric nature.

*Keywords:* algebraic quantum field theory; infrared problem; charge classes; statistics; Lorentz covariance; energy–momentum spectrum

## 1 Outline

The understanding of the sector structure in quantum field theories with long range forces, such as quantum electrodynamics, is a longstanding problem. Its various aspects have received considerable attention in the past, cf. for example [1–4] and references quoted there. But a fully satisfactory solution has not yet been accomplished. We report here on a novel approach which has been developed in collaborations with Sergio Doplicher and John E. Roberts. It sheds new and promising light on this problem.

Recall that a superselection sector is a subspace of the Hilbert space of all states of finite energy on which the local observables of the theory act irreducibly; hence all superselected charges have sharp values and the superposition principle holds unrestrictedly in each sector. The presence of long range forces leads to an abundance of different sectors due to the multifarious formation of clouds of low energy massless particles; in fact their comprehensive analysis and classification is unfeasible. In the treatment of models this problem is frequently circumvented by some *ad*

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*hoc* selection of sectors, *e.g.* by choosing a specific physical gauge, and by an inclusive treatment of undetected low energy massless particles. This strategy is a meaningful workaround but it is clearly not a satisfactory conceptual solution of the problem.

In contrast, the sector structure is fully understood in theories describing only massive particles [5–7]. In these theories the sectors are in one-to-one correspondence to the dual of some compact group which is interpreted as global gauge group. Each sector has definite (para-Bose or para-Fermi) statistics and there always exist (cone) localized Bose and Fermi fields, respectively, which transform as tensors under the action of the gauge group and create the sectors from the vacuum state. Moreover, in spite of the possible non-locality of these fields, the spin-statistics theorem and the existence of collision states have been established in these theories. The arguments underlying these results fail, however, in the presence of long range forces [2].

Our resolution of this problem is based on the insight that the arrow of time enters in a fundamental way in the interpretation of the microscopic theory: since it is impossible to perform measurements in the past, the notion of superselection sector becomes physically meaningless in the presence of massless particles. As will be explained, it has to be replaced by the notion of charge class which is based on a natural equivalence relation between sectors [2]. We will focus attention here on the family of simple charge classes which covers the electric charge. It can be shown that there is a physically meaningful way to assign to each member of this family definite (Bose or Fermi) statistics, there always exists a corresponding conjugate charge class with the same statistics and the family of all simple charge classes determines a compact abelian gauge group. Moreover, there is a natural action of the Poincaré group on each charge class which is implemented by a unitary representation satisfying the relativistic spectrum condition. The generators of the time translations, however, do not have the familiar interpretation as global energy. In fact, they resemble the Liouvillians in Quantum Statistical Mechanics since they subsume in a gross manner energetic fluctuations of the infrared background.

## 2 Input

The present approach applies to theories of local observables fitting into the general algebraic framework of quantum field theory [4]. Any such theory provides an assignment (net)  $\mathcal{R} \mapsto \mathcal{A}(\mathcal{R})$  mapping spacetime regions  $\mathcal{R} \subset \mathbb{R}^4$  to unital  $C^*$ -algebras  $\mathcal{A}(\mathcal{R})$  generated by the observables localized in the respective regions. The  $C^*$ -inductive limit of these local algebras is denoted by  $\mathcal{A}$  and assumed to act on the defining vacuum Hilbert space  $\mathcal{H}$  of the theory. The net satisfies the condition of locality (spacelike commutativity)

$$[\mathcal{A}(\mathcal{R}_1), \mathcal{A}(\mathcal{R}_2)] = 0 \quad \text{if } \mathcal{R}_1 \times \mathcal{R}_2$$

and of covariant automorphic action  $\alpha : \mathcal{P}_+^\uparrow \rightarrow \text{Aut } \mathcal{A}$  of the Poincaré group  $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$ ,

$$\alpha_\lambda \mathcal{A}(\mathcal{R}) = \mathcal{A}(\lambda \mathcal{R}), \quad \lambda \in \mathcal{P}_+^\uparrow.$$

There is an, up to a phase unique, vector state  $\Omega \in \mathcal{H}$  describing the vacuum and fixing a continuous unitary representation  $U : \mathcal{P}_+^\uparrow \longrightarrow \mathcal{U}(\mathcal{H})$  through the relation

$$U(\lambda) A \Omega = \alpha_\lambda(A) \Omega, \quad \lambda \in \mathcal{P}_+^\uparrow, \quad A \in \mathcal{A}.$$

The subgroup of spacetime translations satisfies the spectrum condition,  $\text{sp } U \upharpoonright \mathbb{R}^4 \subset \overline{V}_+$ . Thinking primarily of theories describing interactions of electromagnetic type, we assume that there exist massless single particle states in  $\mathcal{H}$  (photons) and there is some mass threshold above which there appear pairs of massive particles carrying opposite charges (electron–positron pairs *etc*), cf. Fig. 1. The scattering states of photons are assumed to be asymptotically complete below this threshold [2].

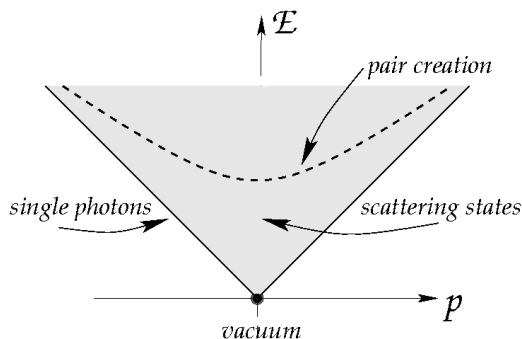


Figure 1: Energy–momentum spectrum in vacuum sector

Besides the states in the vacuum sector there exist other elementary states of physical interest, describing charged particles, atoms, ions, molecules, *etc*. We adopt here the convenient point of view that these states are described by suitable representations  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  acting all on the same (separable) Hilbert space  $\mathcal{H}$ . The (identical) vacuum representation is denoted by  $\iota$ .

**Selection criterion for states of interest:** The states of elementary systems are described by irreducible representations  $(\pi, \mathcal{H})$  of  $\mathcal{A}$  for which there is a continuous unitary representation  $U_\pi : \mathbb{R}^4 \rightarrow \mathcal{U}(\mathcal{H})$  such that  $\text{Ad } U_\pi(x) \circ \pi = \pi \circ \alpha_x$  (covariance) and  $\text{sp } U_\pi \subset \overline{V}_+$  (stability).

Note that it is not required in this criterion that the Lorentz transformations are also implemented. Because this would exclude from the outset states carrying an electric charge [8]. On the other hand there is an abundance of disjoint representations satisfying this criterion which, however, differ only by unobservable infrared properties as will be explained next.

### 3 Charge classes

Realistic experiments are performed in finite spacetime regions. Beginning at some appropriate spacetime point one performs preparations of states and measurements until sufficient data are taken. In principle, subsequent generations of experimentalists could continue the experiment into the distant future. Thus the maximal regions where data can be taken are future directed lightcones  $V$  with arbitrary apex  $(t_0, x_0)$ , cf. Fig. 2. On the other hand it is impossible to make up

for missed operations in the past of the initial point  $(t_0, x_0)$ . Hence, as a consequence of the “arrow of time”, it suffices for the comparison of theory and experiment to consider the restrictions of global states to the algebra  $\mathcal{A}(V)$  of observables localized in any given future directed lightcone  $V$ . These restrictions are called partial states.

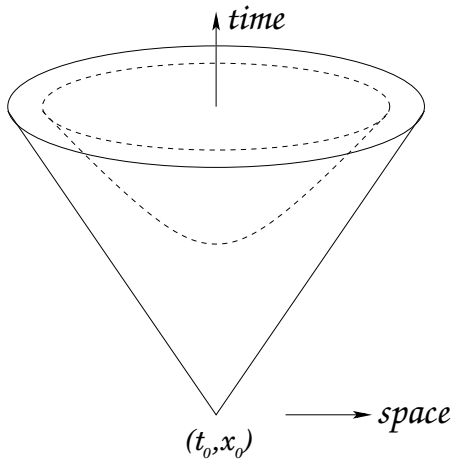


Figure 2: Spacetime region  $V$  foliated with a hyperboloid (time shell)

In massive theories the algebras  $\mathcal{A}(V)$  are irreducible [9] and complete information about the underlying global states can be obtained in any lightcone  $V$ . The situation is markedly different, however, in the presence of massless particles [2]. This is so because outgoing radiation created in the past of  $(t_0, x_0)$  has no observational effects in  $V$  in accordance with Huygens’ principle. As a consequence, infrared clouds cannot sharply be discriminated by measurements in any given lightcone  $V$  and the algebras  $\mathcal{A}(V)$  are highly reducible; in fact, their weak closures  $\mathcal{A}(V)^-$  are factors of type  $\text{III}_1$ . Whereas the infrared sectors cannot be distinguished in any lightcone  $V$ , their total charge can be determined there. This follows from the fact that the charge is tied to massive particles which eventually enter  $V$ , unless they are annihilated in pairs carrying opposite charges. These heuristic considerations suggest to introduce the following equivalence relation [2].

**Definition of charge classes:** Let  $(\pi_1, \mathcal{H})$ ,  $(\pi_2, \mathcal{H})$  be representations satisfying the above selection criterion and let  $V$  be any lightcone. The representations belong to the same charge class if their restrictions to  $\mathcal{A}(V)$  are unitarily equivalent,

$$\pi_1 \upharpoonright \mathcal{A}(V) \simeq \pi_2 \upharpoonright \mathcal{A}(V).$$

It can be shown that the charge classes do not depend on the choice of  $V$ ; moreover, the restricted representations are primary, *i.e.* all charges which can be determined in  $V$  have sharp values within a given charge class. We also note that the restricted representations can be reconstructed from the partial states. Thus only data which can be taken in  $V$  are needed in order to fix the charge classes. We therefore propose to replace the notion of superselection sector by the (in the presence of massless particles more realistic) concept of charge class.

## 4 Charged morphisms

In order to clarify the structure of the family of charge classes one has to understand their mutual relation. Given  $V$ , one can proceed from the partial states in the charge class of the vacuum to the partial states in a given charge class by limits of local operations in  $V$ . Heuristically, these operations may be thought of as creation of pairs of opposite charges on some given time shell, where the unwanted compensating charge is shifted to lightlike infinity; it thereby disappears in the spacelike complement of any relatively compact region in  $V$ . In order to control the energy required for these operations one has to localize them in broadening hypercones  $\mathcal{L}$ , cf. Fig. 3. (A hypercone is the causal completion of an open pointed convex cone formed by geodesics on some time shell in  $V$ .) These heuristic considerations are put into mathematically precise form as follows.

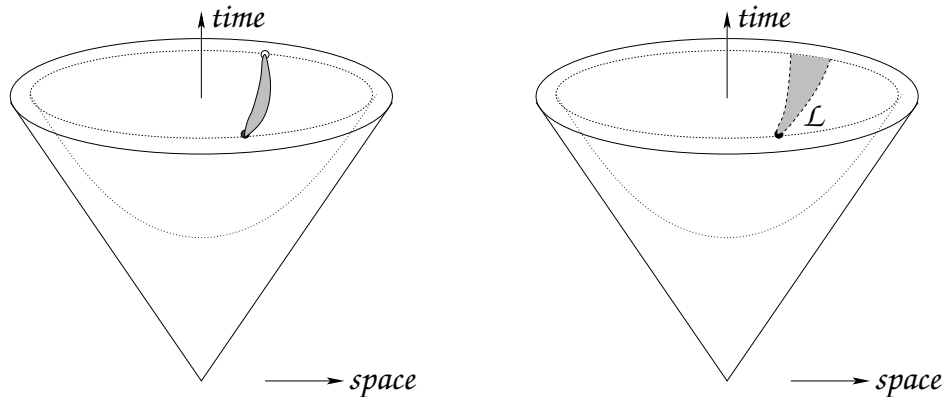


Figure 3: Local operation creating a pair and charge creation in a hypercone  $\mathcal{L}$  as a limit case

**Assumption:** Given a charge class, there exists for any hypercone  $\mathcal{L} \subset V$  a sequence of inner automorphisms  $\{\sigma_n \in \text{In } \mathcal{A}(\mathcal{L})\}_{n \in \mathbb{N}}$ , induced by unitaries in  $\mathcal{A}(\mathcal{L})$ , such that

$$\rho_{\mathcal{L}} \doteq \text{w-}\lim_n \sigma_n \quad \text{exists pointwise on } \mathcal{A}(V)$$

and the adjoint of  $\rho_{\mathcal{L}}$  maps the partial states in the charge class of the vacuum onto the partial states in the target charge class.

The properties of the limit maps are summarized in the following proposition.

**Proposition 4.1.** *For fixed charge class and any  $\mathcal{L} \subset V$ , let  $\rho_{\mathcal{L}} : \mathcal{A}(V) \rightarrow \mathcal{A}(V)^-$  be the map defined above.*

- (a)  $\rho_{\mathcal{L}}$  is linear, symmetric and multiplicative
- (b)  $\rho_{\mathcal{L}} \upharpoonright \mathcal{A}(\mathcal{R}) = \iota$  if  $\mathcal{R} \times \mathcal{L}$
- (c)  $\rho_{\mathcal{L}}(\mathcal{A}(\mathcal{R}))^- \subseteq \mathcal{A}(\mathcal{R})^-$  if  $\mathcal{R} \supseteq \mathcal{L}$ .
- (d)  $\rho_{\mathcal{L}_1} \simeq \rho_{\mathcal{L}_2}$  for any pair of hypercones  $\mathcal{L}_1, \mathcal{L}_2 \subset V$ .

According to point (a) these maps define representations of the algebra  $\mathcal{A}(V)$  for the given lightcone  $V$ . Points (b) and (c) encode the information that they arise from local operations in  $\mathcal{L}$  and point (d) expresses the fact that infrared clouds, which are inevitably produced by the charge creating operations, cannot sharply be discriminated in  $V$ . In analogy to the terminology used in sector analysis, these maps are called (hypercone) localized morphisms.

We restrict attention here to the simplest, but physically important family of charge classes where in part (c) of the proposition equality holds for the corresponding morphisms. Then  $\rho_{\mathcal{L}}(\mathcal{A}(V))^- = \mathcal{A}(V)^-$  is a factor of type  $\text{III}_1$ . Taking the fact into account that the space of normal states on such factors is homogeneous [10], *i.e.* the (adjoint) inner automorphisms of  $\mathcal{A}(V)^-$  act almost transitively on normal states, it is meaningful to assume that there exist morphisms as in point (d) of the proposition which are related by unitary intertwiners in  $\mathcal{A}(V)^-$ . We summarize these features in the following definition.

**Definition of simple charge classes:** A charge class is said to be simple if for any hypercone  $\mathcal{L} \subset V$  there exist corresponding localized morphisms  $\rho_{\mathcal{L}}$  such that

- (i)  $\rho_{\mathcal{L}}(\mathcal{A}(\mathcal{R}))^- = \mathcal{A}(\mathcal{R})^-$  if  $\mathcal{R} \supseteq \mathcal{L}$
- (ii)  $\rho_{\mathcal{L}_1} \simeq \rho_{\mathcal{L}_2}$  for any pair of hypercones  $\mathcal{L}_1, \mathcal{L}_2 \subset V$  and there exist corresponding unitary intertwiners in  $\mathcal{A}(V)^-$ .

Localized morphisms attached to a simple charge class are said to be simple.

## 5 Statistics and symmetries

Similarly to the case of superselection sectors, one has to rely in the analysis of charge classes on a maximality condition for the hypercone algebras, akin to Haag duality: for any hypercone  $\mathcal{L} \subset V$  one has

$$\mathcal{A}(\mathcal{L})' \cap \mathcal{A}(V)^- = \mathcal{A}(\mathcal{L}^c)^-, \quad \mathcal{A}(\mathcal{L}^c)' \cap \mathcal{A}(V)^- = \mathcal{A}(\mathcal{L})^-,$$

where  $\mathcal{L}^c$  denotes the spacelike complement of  $\mathcal{L}$  in  $V$ . Substantial results supporting this form of hypercone duality have been established in [11] for the free Maxwell field.

It is an important consequence of hypercone duality that equivalent morphisms which are localized in neighboring hypercones  $\mathcal{L}_1, \mathcal{L}_2$  have intertwiners which are contained in  $\mathcal{A}(\mathcal{L})^-$ , where  $\mathcal{L}$  is any larger hypercone containing  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Making use of this fact one can extend the morphisms from their domain  $\mathcal{A}(V)$  to larger algebras containing the weak closures of certain hypercone algebras. Based on these extensions, the following result for the family of simple localized morphisms has been established.

**Proposition 5.1.** *Let  $\rho_1, \rho_2$  be simple morphisms which are localized in hypercones  $\mathcal{L}_1, \mathcal{L}_2$ , respectively.*

- (a) *The (suitably extended) morphisms can be composed and there is for any given hypercone  $\mathcal{L}$  some simple morphism  $\rho$  localized in  $\mathcal{L}$  such that  $\rho_1 \circ \rho_2 \simeq \rho$ .*

- (b)  $\rho_1 \circ \rho_2 \simeq \rho_2 \circ \rho_1$ . If  $\rho_1, \rho_2$  belong to the same charge class there is some canonical intertwiner  $\varepsilon(\rho_1, \rho_2) \in \mathcal{A}(V)^-$  depending only on the given morphisms.
- (c) For each charge class there exists a statistics parameter  $\varepsilon \in \{\pm 1\}$  such that for any pair of morphisms  $\rho_1, \rho_2$  in this class which are localized in spacelike separated hypercones  $\mathcal{L}_1, \mathcal{L}_2$ , cf. Fig. 4, one has  $\varepsilon(\rho_1, \rho_2) = \varepsilon 1$ .

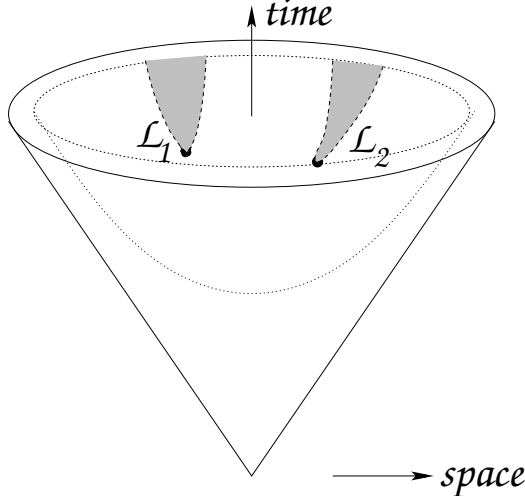


Figure 4: Spacelike separated hypercones

- (d) For each simple charge class there exists a simple conjugate charge class such that for any morphism  $\rho$  in the given class there is a corresponding morphism  $\bar{\rho}$  in the conjugate class satisfying  $\rho \circ \bar{\rho} = \bar{\rho} \circ \rho = \iota$ . Moreover, the conjugate class has the same statistics parameter as the given class.

These results do not depend on the choice of lightcone  $V$ . According to item (c) any simple charge class has definite (Bose respectively Fermi) statistics and item (d) says that to each simple charge class there is a simple conjugate class of states carrying opposite (neutralizing) charges. Moreover, items (a), (b) and (d) imply that the equivalence classes of simple morphisms determine an abelian group with the product given by composition. Its dual can be interpreted as global gauge group generated by the simple charges which can be sharply determined in lightcones. Thus the simple charge classes have a structure similar to that of the simple sectors in massive theories. The elusive theoretical effects of the infrared clouds completely disappear from the discussion by taking into proper account the spacetime limitations of real experiments.

## 6 Covariance and spectrum

In the discussion of the covariance properties of simple charge classes and of their energetic features, one has to take into account that one has merely an endomorphic action of the semigroup  $\mathcal{S}_+^\uparrow \doteq \bar{V}_+ \rtimes \mathcal{L}_+^\uparrow \subset \mathcal{P}_+^\uparrow$  on any lightcone  $V$ . The following characterization of covariant simple morphisms is appropriate in this case.

**Definition of covariant morphisms:** Let  $\rho : \mathfrak{A}(V) \rightarrow \mathfrak{A}(V)^-$  be a simple morphism which is localized in  $\mathcal{L}$ . The morphism is said to be covariant if it is the initial member of a family of morphisms  $\{\lambda\rho\}_{\lambda \in \mathcal{S}_+^\uparrow}$  which, for given  $\lambda \in \mathcal{S}_+^\uparrow$ , are localized in any hypercone  ${}^\lambda\mathcal{L} \supset \lambda\mathcal{L}$ , satisfy the relation

$$\lambda^\mu\rho \circ \alpha_\lambda = \alpha_\lambda \circ \mu\rho, \quad \lambda, \mu \in \mathcal{S}_+^\uparrow$$

and are affiliated with the same charge class as  $\rho \equiv {}^1\rho$ . More precisely, there exists a weakly continuous section of intertwiners  $\lambda \mapsto \Gamma_\lambda \in \mathcal{A}(V)^-$  between  $\lambda\rho$  and  $\rho$ .

It can be shown that the family of morphisms affiliated in this manner with a given morphism is unique. The properties of covariant morphisms are described in the following proposition.

**Proposition 6.1.** *Consider the family of covariant simple morphisms which are localized in hypercones contained in a given lightcone  $V$ .*

- (a) *The family is stable under composition and conjugation and all results of Proposition 5.1 apply to it.*
- (b) *Each morphism  $\rho$  determines a unique unitary representation  $U_\rho$  of (the covering of) the full Poincaré group  $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \tilde{\mathcal{L}}_+^\uparrow$  such that*

$$AdU_\rho(\tilde{\lambda}) \circ \rho = \rho \circ \alpha_\lambda, \quad \tilde{\lambda} \in \tilde{\mathcal{S}}_+^\uparrow$$

- (c)  *$spU_\rho \upharpoonright \mathbb{R}^4 \subset \overline{V}_+$*

- (d) *In presence of massless particles  $U_\rho(\mathbb{R}^4) \notin \mathcal{A}(V)^-$ .*

This result shows that the covariant morphisms describe the expected physical properties of elementary systems in a meaningful way. In particular, it is possible to assign with the help of the unitary groups established in (b) an energy content to the partial states on  $\mathcal{A}(V)$  within a given charge class. This energy is bounded from below according to (c), expressing the stability of the states. But in view of point (d) the generators of the time translations should not be interpreted as genuine observables. This can be understood if one bears in mind that part of the energy content of a global state may be lost by outgoing radiation created in the past of  $V$ . Phrased differently, the energy content of the partial states on  $\mathcal{A}(V)$  is fluctuating. Thus the generators of the time translations in (b) require an interpretation similar to that of the Liouvillians in quantum statistical mechanics.

## 7 Concluding remarks

In this work the origin of the infrared difficulties in the interpretation of theories with long range forces has been traced back to the unreasonable idealization of observations covering all of Minkowski space. Observations are at best performed in future directed lightcones, hence the arrow of time enters already in the interpretation of the microscopic theory.



As was explained, the restriction of states to the observables in any given lightcone  $V$  amounts to a geometric, Lorentz invariant infrared regularization. It allows to form charge classes of states carrying the same global charges but differing in their infrared features. The pertinent observable algebras  $\mathcal{A}(V)$  are highly reducible due to the loss of information about outgoing radiation created in the past of  $V$ . Yet the data which can be obtained in  $V$  are sufficient to determine sharply the global charges, their statistics and the underlying gauge group. These results have been established so far only for simple charge classes; but work in progress indicates that they hold more generally.

The data obtainable in lightcones  $V$  are also sufficient to fix for each charge class a representation  $U_V$  of the Poincaré group, respectively of its covering. Yet the generators of this representation cannot be interpreted as genuine observables since they incorporate in a gross manner fluctuations of the background radiation. In this respect the situation resembles the treatment of ensembles in quantum statistical mechanics.

Since infrared sectors cannot be discriminated in any lightcone  $V$  it seems likely that also the infraparticle problem (failure of the Wigner concept of particle for electrically charged states [1,8]) disappears if one resorts to the representations  $U_V$  of the Poincaré group. In fact, the radiation observed in  $V$  can be described by Fock states with a finite particle number; so there may well exist partial states on  $\mathcal{A}(V)$ , describing a single electrically charged particle where the (globally inevitable) accompanying radiation field has no observable effects in  $V$ . Such states could well contribute to an atomic part in the mass spectrum of  $U_V$ .

## References

- [1] J. Fröhlich, G. Morchio and F. Strocchi, “Charged sectors and scattering states in quantum electrodynamics”, *Annals Phys.* **119** (1979) 241
- [2] D. Buchholz, “The physical state space of quantum electrodynamics”, *Commun. Math. Phys.* **85** (1982) 49
- [3] O. Steinmann, *Perturbative Quantum Electrodynamics and Axiomatic Field Theory*, Berlin, Springer (2000)
- [4] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Berlin, Springer (1992)
- [5] S. Doplicher, R. Haag and J. E. Roberts,  
“Local observables and particle statistics 1”, *Commun. Math. Phys.* **23** (1971) 199.  
“Local observables and particle statistics 2”, *Commun. Math. Phys.* **35** (1974) 49.
- [6] D. Buchholz and K. Fredenhagen, “Locality and the structure of particle states”, *Commun. Math. Phys.* **84** (1982) 1
- [7] S. Doplicher and J. E. Roberts, “Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics”, *Commun. Math. Phys.* **131** (1990) 51
- [8] D. Buchholz, “Gauss’ law and the infraparticle problem”, *Phys. Lett. B* **174** (1986) 331
- [9] P. Sadowski and S. L. Woronowicz, “Total sets in quantum field theory”, *Rept. Math. Phys.* **2** (1971) 113
- [10] A. Connes and E. Størmer, “Homogeneity of the state spaces of factors of type III<sub>1</sub>”, *J. Funct. Anal.* **28** (1978) 187
- [11] P. Camassa, “Relative Haag duality for the free field in Fock representation”, *Annales Henri Poincaré* **8** (2007) 1433